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Computing and interpreting the adjacency spectrum of traffic networks

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Abstract: It is investigated whether certain structural properties of a traffic network can be identified by an analysis of the spectrum of its adjacency matrix

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1. Introduction

In the geographical literature there are several papers dealing with the problem of deriving information about a traffic network by analyzing the spectrum of matrices connected with it [2,5,7,8,9]. In this paper several ideas for the interpretation of the spectrum of the adjacency matrix are presented, among them a strategy for the identification of subsets of vertices that are better connected internally than with the rest of the network.

In [5] and [7] one of the standard examples is the Uganda traffic network of 1921. Although it is a tree and therefore a rather special graph, it is quite useful for illustrating the topics of the following paragraphs. The spectrum of its adjacency matrix is given in the appendix.

At the end of this article there are some considerations concerning the numerical problems that arise during the computation of such a spectrum.

Conventions: The vertices of a graph will be named $x_1, \ldots, x_n$. λ is always an eigenvalue and $v = (v_1, \ldots, v_n)^T$ an eigenvector. A path with n vertices (and $n-1$ edges) will be called W_n.

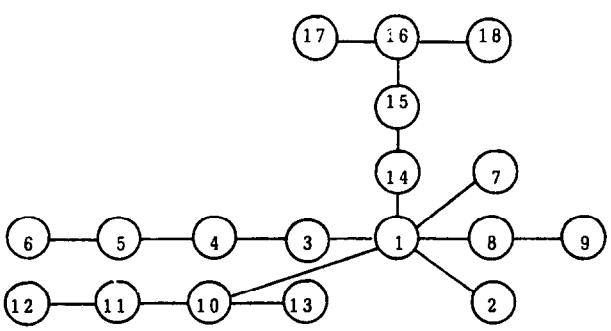

Fig. 1. Uganda road traffic network 1921 (the towns are numbered according to Gould [5], vertex 1 is the capital Kampala).

2. The spectral radius

As the adjacency matrix A of a connected undirected graph G is symmetrical, nonnegative and irreducible, $\lambda_1 := \rho(A)$ is a simple eigenvalue of A and there is a positive eigenvector $v^{(1)}$ corresponding to λ_1. Obviously one can introduce an ordering of the vertices of G letting a vertex x_i rank the higher the higher the corresponding entry $v_i^{(1)}$ is. (Of course, a real distinction between the vertices is obtained this way, only if G is not regular.) It is very suggestive to say that a vertex ranking high in this ordering has a 'strong position' in the network. Furthermore, as $\rho(A)$ increases (decreases), if an edge is inserted into G (deleted from G), one should expect $\rho(A)$ (or the ratio $\rho(A)/(n-1)$ between the actual spectral radius and the maximal attainable one for networks with n vertices) to give some information about how 'good' or 'bad' the connections in the network are. This expectation is expressed by Cliff, Haggett and Ord [2] by claiming that $\rho(A)$ should be a good measure for the 'overall connectedness' of a network and that the abovementioned ordering ranks the vertices according to how well connected they are. The problem with this interpretation is that there is no previous definition of well-connectedness by which these statements could be tested. A straightforward definition could be that a network is well connected, if you can get from everywhere to everywhere else quickly (which more or less means that its diameter is small). It is not too difficult to accept this definition in the light of the data from the Uganda 1921 network.

But it is also easy to give examples of graphs where a better overall connectedness in this sense does not coincide with a higher spectral radius (e.g. linking the endvertices of a path W_n by an edge yields $\rho = 2$, whereas a new edge between any two of the other vertices yields $\rho > 2$). This is due to the fact that the spectral radius honors more a new link between two vertices with a high rank than between two vertices having a low rank up to now.

A possible way out of this problem might be just to take the data from $\rho(A)$ and $v^{(1)}$ as a *definition* of well-connectedness. For real-world traffic networks this could be sensible: Generally, places from where you can get to other places quickly attract people so that their number of inhabitants is greater than that of other settlements. A new connection between two high-ranking towns is then indeed a greater achievement than a connection between two low-ranking ones, because more people profit directly from this improvement. (Of course this method is not without problems when applied to subgraphs of the network.)

3. Symmetry

Symmetries that are present in a graph have a direct influence on its spectrum. Halin [6] has listed several results about this connection. E.g. if for two vertices x_i, x_j there is an automorphism mapping them onto each other, then in the eigenvector $v^{(1)}$ there must be $v_i^{(1)} = v_j^{(1)}$ and in an eigenvector $v^{(k)}$, $k > 1$, belonging to a simple eigenvalue $v_i^{(k)}$ and $v_j^{(k)}$ can only differ in their sign. If the vertices x_i and x_j are interchangeable, i.e. if there is an automorphism P with

$$P(x_k) = \begin{cases} x_i, & \text{if} \quad k = j, \\ x_j, & \text{if} \quad k = i, \\ x_k & \text{otherwise,} \end{cases}$$

then the vector v having 1 in the ith place and -1 in the jth place and zeros everywhere else is an eigenvector of the adjacency matrix of the graph (and of certain other matrices connected with the graph too) and the corresponding eigenvalue is zero if x_i and x_j are not connected and -1 otherwise. In the Uganda network this is the case for the pairs of vertices x_2, x_7 and x_{17}, x_{18}.

Looking closely at the Uganda graph we see that there is a sort of "symmetry in a weak sense" which has nothing to do with a graph automorphism. The vertices $\{x_3, x_4, x_5, x_6\}$ and $\{x_{10}, x_{11}, x_{12}, x_{13}\}$ each form a copy of the graph W_4 and both are connected with vertex x_1 as the only 'external' neighbour. Of course this is reflected in the spectrum by the fact that all eigenvalues of W_4 are also eigenvalues of the Uganda graph with corresponding eigenvectors that leave all entries at zero that do not belong to vertices from a copy of W_4.

Furthermore we see that for all other eigenvectors $v^{(k)}$ (with respect to numerical accuracy) we have $v_4^{(k)} \approx v_{13}^{(k)}$ and $v_5^{(k)} \approx v_{12}^{(k)}$. The following considerations show in a slightly more general context that this is no mere coincidence but a consequence of the fact that two paths of the same length have the same connection to the 'external world'.

Let the path W_n with vertices $x_1, \ldots, x_n$ be connected with the rest of the graph by just one edge connecting x_k with a vertex y. Let λ be an eigenvalue of the graph which is no eigenvalue of W_n. Let v be a corresponding eigenvector of the graph with entries $v_1, \ldots, v_n$ corresponding to $x_1, \ldots, x_n$ and entry $-\alpha \neq 0$ belonging to y. The numbers $v_1, \ldots, v_n$ then must solve the following equations:

$$\begin{bmatrix} -\lambda & 1 & 0 & & \ldots & 0 \\ 1 & -\lambda & 1 & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & \ldots & 0 & 1 & -\lambda & 1 \\ 0 & \ldots & & 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is regular and on the right side there is α in the kth row.

Case 1. $k = n$

Let $N(\lambda)$ be a polynomial in λ that will be determined later. Let $a_1 := \alpha/N(\lambda)$, $a_2 := \lambda\alpha/N(\lambda)$, $a_{i+1} := \lambda a_i - a_{i-1}$, $i = 2, \ldots, n-1$. It is clear that $v_i := a_i$, $i = 1, \ldots, n$ solve the first $n-1$ equations. Furthermore we have $a_{n-1} - \lambda a_n = P(\lambda)\alpha/N(\lambda)$ with $P(\lambda)$ being a polynomial in λ. If $P(\lambda) = 0$, $a_1 := \alpha$, $a_2 := \lambda\alpha$, $a_{i+1} := \lambda a_i - a_{i-1}$ would be a non-trivial solution of the homogeneous system i.e. λ would be an eigenvalue of W_n. Hence $P(\lambda) \neq 0$ and $N(\lambda) := P(\lambda)$ yields a solution of the complete system.

Case 2. $1 < k < n$

Let $b_1, \ldots, b_n$ be the solution of the system. We claim that

(1) $b_1 = a_{n-k+1}$, $b_2 = a_{n-k+1}$, $b_{i+1} = \lambda b_i - b_{i-1}$, $i = 2, \ldots, k-1$,

(2) $b_n = a_k$, $b_{n-1} = \lambda a_k$, $b_{j-1} = \lambda b_j - b_{j+1}$, $j = n-2, \ldots, k+1$.

Obviously the construction under (1) solves the first $k-1$ equations and the one under (2) solves the last $n-k$ equations. It remains the problem that b_k is defined twice and that the kth equation is not yet shown to hold.

As $a_1 \neq 0$, we can write

(1′) $b_i = (a_{n-k+1}/a_1) \cdot a_i, \quad i = 1, \ldots, k,$

(2′) $b_j = (a_k/a_1) \cdot a_{n-j+1}, \quad j = n, \ldots, k.$

In both cases we have $b_k = a_k a_{n-k+1}/a_1$.
We still have to show

$$b_{k-1} - \lambda b_k + b_{k+1} = a_{n-1} - \lambda a_n.$$

Because of (1′) and (2′) this is equivalent to

$$a_{n-k+1}a_{k-1} - \lambda a_k a_{n-k+1} + a_k a_{n-k} = a_1 a_{n-1} - \lambda a_1 a_n \quad \text{for arbitrary} \quad k$$

(without loss of generality $k \geq [\frac{1}{2}n]$).
For $k = n - 1$ this equation can be shown directly.
In general we have

$$\begin{aligned} a_{n-k+1}a_{k-1} - \lambda a_k a_{n-k+1} + a_k a_{n-k} &= a_{n-k+1}a_{k-1} + a_k a_{n-k+2} \\ &= a_{n-k+2}a_{k-2} - \lambda a_{k-1}a_{n-k+2} + a_{k-1}a_{n-k+1} \end{aligned}$$

which is the same expression as the first one just with $k - 1$ replacing k. This finishes the proof that the b-vector is defined correctly. In the Uganda graph we have $n = 4$, $k = 3$ and therefore it must hold that $a_1 = b_4$ and $a_2 = b_1$.

Although there is a clear relationship between symmetries in a graph and its spectrum, these results might be of only limited value for the analysis of traffic networks. Unfortunately quite a few symmetries only arise because the real world is modelled too roughly by the graph thus creating equivalences between vertices that do not exist in reality.

4. Separation of substructures

For orthogonality reasons eigenvectors v belonging to subdominant eigenvalues λ have positive and negative elements (and possibly some zeros). So it has been asked what information about the graph the separation of its vertices in those with positive corresponding entries in v and those with negative corresponding entries in v reveals. Cliff, Haggett and Ord [2] consider the eigenvalue with the second largest absolute value for several airline traffic networks and mention that the components of the eigenvector "seem to delineate distinct regions with between-region links being relatively weak compared to the within-region links". For the Uganda network the eigenvalue in question would be the lowest one $\lambda_{18} = -\rho(A)$ and for every pair of neighbours the corresponding entries in $v^{(18)}$ would have different signs. As this is a general feature for bipartite graphs this interpretation rule can surely not be applied to a wide range of cases.

Tinkler [8] considers the eigenvectors as stable states of an information flow in the graph: for some eigenvector v every vertex having a positive entry in v tells all its neighbours a certain statement. Every vertex having a negative entry in v tells all its neighbours exactly the opposite statement. The intensity with which the statement is told is proportional to the intensity with which this vertex has received the news (with contradictory information of the same intensity eliminating one another).

To this author, however, another interpretation seems more suitable. It originates from Collatz and Sinogowitz [3] for graphs that are a discretization of a membrane and has been extended by Sachs [3, p. 252–257] to general graphs. Here we consider the vertices of a graph as objects of mass one oscillating orthogonally to the surface they are drawn on and connected by edges of length and elasticity one. Additionally, every vertex is connected to so many external non-moving points (by edges of the above type) that the original graph becomes regular of degree r. The eigenvectors then describe the elongations of the vertices during the eigenvibrations of the graph, and from the eigenvalues we get information about the frequency $(=\sqrt{r-\lambda})$ and the energy $(\sim r-\lambda)$ of the vibrations. During these vibrations the greatest part of the potential energy will be contained in the edges connecting vertices that move into opposite directions. Therefore let us define $\mu_i := r-\lambda_i$ to be the *separation energy* for the classes of vertices with positive and negative entries in $v^{(i)}$. In order to find regions "with between-regions links being relatively weak compared to the within-region links" [2, p. 314] we shall look for subsets of vertices the separation of which from the rest of the graph does not need a high separation energy.

For the Uganda 1921 network we see that the lowest separation energy is needed by the vertices $x_{14},\ldots,x_{18}$ and the second lowest one by $x_3,\ldots,x_6$ and $x_{10},\ldots,x_{13}$ respectively. On the other hand, only the vibrations with high energy separate the vertices so that x_1 and its neighbours of valency 1, x_2 and x_7 belong to different classes. The highest energy level is achieved of course by the vibration for which every edge connects vertices vibrating in opposite directions.

It is perhaps clear that these considerations do not give an undisputable meaning to *every* eigenvector and in general some more investigations are needed about to what extent the inspection of certain spectra is useful for the analysis (and possibly also for the synthesis) of traffic networks.

5. Numerical aspects

For the actual computation of eigenvalues and eigenvectors Cliff and Ord [1] recommend the v.Mises–Hotelling iteration method. As only a brief description of the algorithm itself shall be given here, the reader is referred to [10] for proofs and references to the original literature on this method. For the analysis of an arbitrary matrix A the algorithm consists of two parts:

1. v.Mises iteration (= power method)

Starting with an arbitrary vector $w^{(0)}$ compute the sequence

$$w^{(i+1)} := Aw^{(i)}/\|Aw^{(i)}\|_p, \quad i=1,2,\ldots$$

($\|\cdot\|_p$ is an arbitrary vector norm). If this sequence convergences, $\lambda_1 := \lim_{i\to\infty}\|Aw^{(i)}\|_p$ is the dominant eigenvalue of A and $v^{(1)} := \lim_{i\to\infty} w^{(i)}$ is a corresponding eigenvector.

2. Hotelling's deflation

If λ_1 is the dominant eigenvalue of A, $Av^{(1)}=\lambda_1 v^{(1)}$ and $A^{\mathrm{T}}y^{(1)}=\lambda_1 y^{(1)}$, form the matrix

$$A^{(1)} := A - \lambda_1 \frac{y^{(1)}v^{(1)\mathrm{T}}}{y^{(1)\mathrm{T}}v^{(1)}}$$

$A^{(1)}$ keeps all right (left) eigenvectors of A that are orthogonal to $v^{(1)}$ ($y^{(1)}$) and, except from λ_1, it also has the same eigenvalues as A, but $v^{(1)}$ and $y^{(1)}$ are eigenvectors of $A^{(1)}$ and $A^{(1)T}$ respectively corresponding to the eigenvalue zero. The dominant eigenvalue of $A^{(1)}$ that the power method will determine is thus the eigenvalue of A with second largest absolute value.
If the spectrum of $A^{(1)}$ is deflated as well by constructing a matrix $A^{(2)}$, this matrix' dominant eigenvalue is the eigenvalue of A with third largest absolute value etc.

In order to make this procedure actually work there are several problems to be overcome:
(i) The deflation step cannot work, if $y^{(1)}$ is orthogonal to $v^{(1)}$ as this is e.g. the case for

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad |\alpha| < 2.$$

(ii) The power method fails, if there are several different dominant eigenvalues (e.g. if A is the adjacency matrix of a bipartite graph).

(iii) Eigenvectors of A corresponding to the eigenvalue zero cannot be found by this method (if zero is a k-fold eigenvalue we have $A^{(n-k)} = \cdots = A^{(n-1)} = 0$).

(iv) The deflation of the spectrum of A by constructing the matrices $A^{(i)} (i = 1, \ldots, n-1)$ only works properly, if in each step λ, y and v are given with their exact numerical values. As in the computational practice this condition cannot be fulfilled, we must be aware of a certain influence of rounding-errors increasing with every deflation step.

In the present paper we only deal with symmetrical matrices. Therefore difficulty (i) cannot occur. Because of this symmetry difficulty (ii) can be avoided by adding a high enough number to the main diagonal of A so that its spectrum becomes positive.

This at the same time avoids difficulty (iii). The last difficulty, however, remains a serious threat to the usefulness of the whole method. A rough check for the correctness of the eigenvalues would be to calculate the deviation of their sum from the trace of A. Furthermore, the inverse power method enables us to determine better values for the eigenvalues starting from the numbers $\lambda_1, \ldots, \lambda_n$ computed so far.

When computing the eigenvalues and eigenvectors of the Uganda traffic network it seemed though, as if the eigenvectors and not the eigenvalues were the difficult point. Even if at every call of the power method the starting-vector was already correct in the first two decimal places, in the most cases it took several dozen steps to find a vector that could possibly (i.e. compared with theoretical results) be correct in the first four digits.

6. Vertex degrees and the first eigenvector

It seems to have always been a temptation to conjecture a certain relationship between the vertex degrees and the entries in the first eigenvector:
– In the Uganda traffic network Gould [5, p.67] investigates several neighbours of the vertex x_1 and states (which is correct in this case) that their corresponding entry in the first eigenvector is higher, if they have a greater valency.
– Tinkler [8, p. 25] writes "It is not unlikely that for a wide variation in point nodality [= vertex degrees, C.M.] the simple nodality of a point will produce a reasonable approximation to the

Table 1

	Eigenvalues									
	$\lambda_1 = 2.652$	$\lambda_2 = 1.828$	$\lambda_3 = 1.618$	$\lambda_4 = 1.447$	$\lambda_5 = 0.9170$	$\lambda_6 = 0.7035$	$\lambda_7 = 0.6180$	$\lambda_8 = 0.4419$	$\lambda_9 = 0$	$\lambda_{10} = 0$
x_1	1	0.11	0	0.43	0.16	0.15	0	0.442	0	0
* x_2	0.38	0.06	0	0.30	0.17	0.21	0	1	1	0
* x_3	0.45	0.123	0.62	−0.06	0.21	0.66	0.62	−0.78	0	0
x_4	0.21	0.117	1	−0.52	0.031	0.32	0.38	−0.79	0	0
* x_5	0.09	0.09	1	−0.69	−0.18	−0.44	−0.38	−0.43	0	0
x_6	0.03	0.05	0.62	−0.48	−0.20	−0.63	−0.62	−0.98	0	0
* x_7	0.38	0.06	0	0.30	0.17	0.21	0	1	−1	0
* x_8	0.44	0.08	0	0.57	−0.92	−0.209	0	−0.24	0	0
x_9	0.17	0.046	0	0.39	−1	−0.30	0	−0.55	0	0
* x_{10}	0.54	0.21	−0.62	−0.76	0.029	0.22	−0.62	−0.35	0	0
x_{11}	0.24	0.17	−0.62	−1	−0.16	−0.31	0.62	0.19	0	0
* x_{12}	0.09	0.09	−0.38	−0.69	−0.18	−0.44	1	0.43	0	0
x_{13}	0.21	0.117	−0.38	−0.52	0.031	0.32	−1	−0.79	0	0
* x_{14}	0.46	−0.34	0	0.28	0.48	−1	0	−0.436	0	0
x_{15}	0.22	−0.73	0	−0.02	0.28	−0.85	0	−0.63	0	0
* x_{16}	0.11	−1	0	−0.31	−0.22	0.40	0	0.16	0	0
x_{17}	0.04	−0.55	0	−0.22	−0.24	0.57	0	0.35	0	1
x_{18}	0.04	−0.55	0	−0.22	−0.24	0.57	0	0.35	0	−1

principal eigenvector, and particularly if the rank relationship alone is considered. For the more restricted nodality of a planar net the relation may be less close ..., but it is implied by Gould for the Uganda 1921 network."
– Cliff, Haggett and Ord [2, p. 311] finally state it as a definite fact "that the components of the first eigenvector are directly proportional to the sum of the entries in the corresponding row or column of the matrix under consideration, and that the eigenvalue is then directly proportional to the largest row or column sum".

A quick glance at the first eigenvector of the Uganda network tells us however that there are quite a lot of vertices having a higher entry in the vector than some other ones with a higher valency.

Appendix

Eigenvalues and eigenvectors of the Uganda network (see Table 1). The eigenvalues $\lambda_{11}, \ldots, \lambda_{18}$ are given by $\lambda_j = -\lambda_{18-j+1}$, and the corresponding eigenvectors are obtained by inverting the sign in the starred rows. (The precision for the eigenvectors is two digits. A third digit has only been given to prevent the impression that certain elements of the same vector were equal.)

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